JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 25, No. 2, March–April 2002

Generalized Active Control of Vibrations in Helicopters

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The problem of active control of vibrations in the main rotor of helicopters is dealt with in the individual blade control framework. We propose a controller valid for a generic flight condition by resorting to a linear parametrically varying (LPV) control technique coupled with the so-called internal model control (IMC) principle. The problem can be reformulated as the one of rejecting a periodic disturbance having known frequency acting on the output of a suitable linear time-varying dynamic model. The main advantage of our control technique is the flexibility of the LPV–IMC approach: It not only can be used with the generic flight condition, but it also can be adapted to comply with a rotor blade model having a complex structure. We obtain a unique controller that automatically gets adapted to the measured advance ratio. An alternative to this control rationale would be to resort to the gain scheduling of periodic controllers, one for each forward flight condition. However, as indicated by our previous research, this would require keeping memory of a large number of control laws in the onboard computer. Furthermore, an H_∞ control strategy is considered to take into account the effects of both unmodeled dynamics and unmodeled disturbances.

Nomenclature

A, B, C, D	=	matrices of the state-space representation of the blade model
d	=	input equivalent disturbance
$d^{(k)}$	=	harmonic at $k\Omega$ in signal d
$\tilde{d}$	=	output disturbance
$\tilde{d}^{(k)}$	=	harmonic at $k\Omega$ in signal $\tilde{d}$
N	=	number of rotor blades
R	=	rotor radius
u	=	control input
V	=	aircraft velocity
v, w	=	process and measurement noise, respectively
W, H	=	state-space matrices for the output noise model
$\boldsymbol{x}$	=	state vector for the input disturbance model
$\tilde{\boldsymbol{x}}$	=	state vector for the output disturbance model
$\boldsymbol{y}$	=	measured output
z	=	noise input of the model for the output disturbance $\tilde{d}$
μ	=	rotor advance ratio
$\boldsymbol{\xi}$	=	state vector for the model of the output disturbance $\tilde{d}$
$\boldsymbol{\xi}^{(k)}$	=	state vector for the model for the kth harmonic of the output disturbance $\tilde{d}^{(k)}$
Ω	=	rotor angular frequency

I. Introduction

IN the recent years, the rejection problem of vibrations in helicopters has received notable attention (e.g., see Refs. 1–9). Besides improving the comfort of the crew and passengers, such an attenuation would be profitable to reduce fatigue in the rotor and structure of the aircraft and to protect onboard equipment from damage. The frequency of the vibration signal induced by the rotation of the blades can be considered constant. Moreover, this frequency is a priori known, and given by $N\Omega$. Hence, our active control design problem is formulated as the problem of attenuating a periodic disturbance (of known frequency).

Among the various approaches proposed in the literature (see Refs. 1 and 8), three main lines of thought can be identified: 1) control of the vibratory response of the airframe by acting on the airframe, 2) control of the vibratory response of the airframe by acting on the main rotor (the so-called higher harmonic control technique), and 3) rotor control by measurement of the vibratory accelerations/loads on each rotor blade individually.

This paper deals with this last strategy, leading to the individual blade control (IBC) method (see Refs. 1 and 9–11 and the references therein). It is generally agreed that the IBC technique is promising. Its consideration is made difficult by the impossibility of resorting to the multiblade transformation so that one must deal with the rotating reference frame. This means that, to cope with the generic flight condition, the appropriate model for control design is necessarily time varying (see Refs. 8, 9, and 12–14). In the past years, our research activity focused on specific flight conditions. In hover, the model turns out to be linear and time invariant so that the control design task can rely on consolidated control techniques, such as pole assignment[7] and the optimal linear quadratic Gaussian (LQG) method.[6] In forward flight (at constant velocity), the situation is more complex because the mathematical model is still linear but is no longer time invariant. Precisely, the system matrices exhibit time periodicity with period π/Ω and with characteristics depending on the aircraft velocity (e.g., see Refs. 12, 15, and 16). This is why a periodic approach has been proposed in Refs. 5 and 17. This approach exhibits good performances when the helicopter is in a forward flight condition and the controller used has been synthesized for the correct velocity. On the contrary, a controller designed for a given velocity may lead to poor performances when the helicopter is in forward flight at a different (constant) velocity.[5] To design a control system able to cope with any time variability in the flight velocity, it is possible to calculate a number of periodic controllers, each one corresponding to a different velocity and using them in a gain-scheduling fashion. This can be impractical because it would be necessary to keep memory of a large number of control laws in the onboard computer.

In this paper, we investigate the applicability and limitations of linear parametrically varying (LPV) control techniques, in their linear matrix inequality (LMI) version e.g., (see Refs. 18–20) for the generic flight case.

In the classical robust control approaches, the typical objective is to design a control law valid for all values of an unknown parameter. This may lead to very conservative control systems. The LPV technique is more flexible because the controller is tuned according to a parameter that is a priori uncertain but that can be measured

Received 1 June 2000; revision received 19 January 2001; accepted for publication 20 January 2001.

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online. In this way, the adaptation of the controller is automatically performed in function of a parameter having a specific physical meaning. This is one of the reasons behind the success of the LPV approach in the recent years in aeronautics[21–23] and in other industrial fields.[24]

The research activity reported herein has been developed in cooperation with a major Italian helicopter company and already has led to the development of simulation codes and identified models for helicopter rotors.[7,15]

The paper is organized as follows. In Sec. II, a precise formulation of the disturbance rejection problem is given and the adopted mathematical model of the rotor blade is briefly introduced. Then, in Sec. III, we present the linear fractional representation of the rotor blade model. This representation is a fundamental step for control design purposes (Sec. IV). Finally, some significant validation experiments are presented in Sec. V. The paper is complemented with two Appendices, the first concerning linear fractional representations and the second concerning the harmonic transfer function, a tool for the solution of the disturbance rejection problem.

As usual, $M > 0\,(M < 0)$ means that the symmetric matrix M is positive (negative) definite. Moreover, the symbol I_r is used to indicate an identity matrix of dimension r. The L_2 norm of a vector valued function $x(\cdot)$ is

$$\|x\|_{L_2} := \left\{ \int_{t=0}^{\infty} x(t)^T x(t)\,\mathrm{d}t \right\}^{\frac{1}{2}}$$

The symbol $\mathcal{F}(\cdot\,,\cdot)$ is used to indicate the lower linear fractional transformation.[25,26] Precisely, given five matrices M_{11}, M_{12}, M_{21}, M_{22}, and Θ of suitable dimensions

$$M(\Theta) := \mathcal{F}\left(\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \Theta\right) = M_{11} + M_{12}\Theta(I - M_{22}\Theta)^{-1}M_{21}$$

This expression is named the linear fractional representation (LFR) of the matrix $M(\Theta)$.

II. Problem Statement

A. Model

As discussed in various papers,[1,6] the vibration attenuation problem can be faced by means of a mathematical model such as the following one:

$$\dot{\tilde{\boldsymbol{x}}}(t) = A(t)\tilde{\boldsymbol{x}}(t) + B(t)u(t) + v(t)$$
$$\boldsymbol{y}(t) = C(t)\tilde{\boldsymbol{x}}(t) + D(t)u(t) + \tilde{d}(t) + w(t) \tag{1}$$

where $u(\cdot)$ is the variation to the pitch angle of the blades (IBC input) and $\boldsymbol{y}(\cdot)$ is the vector of measured output. In the model used for this study, $\boldsymbol{y}(\cdot)$ is a scalar measurement of the vertical shear force at the root of the blade. Also, $\tilde{\boldsymbol{x}}(t) \in \Re^n$ is the state vector of the system. The disturbance $\tilde{d}(t)$ is periodic with angular frequency $N\Omega$, where N is the number of blades and Ω is the angular velocity of the rotor, and $v(t)$ and $w(t)$ are introduced to characterize model mismatches and measurement errors.

The rotor blade model here referred to has been used in a number of previous works (e.g., Ref. 5) and is described in detail in Ref. 16 (also see Ref. 12 for the basic modeling issues). Notice that all the model equations are expressed in dimensionless form: The normalization involves masses (normalized with respect to the air density) and lengths (normalized with respect to the rotor radius R) (Ref. 16). The periodic aerodynamic loads have been computed by means of the classical blade element theory assuming a uniform inflow model.[16] The assumption of linearity of the adopted model (1) is subject to validation: Precisely, it is necessary to verify that the designed control system produces a pitch angle variation $u(t)$ whose amplitude is small enough to be compatible with the linearity assumption.

Notwithstanding its simplicity, this model takes into account the main features of interest, in particular, the flexibility of the blade and the dependence of the model dynamics on the considered flight condition. As discussed in Ref. 16, the model complexity depends on the number of out-of-plane bending modes considered in the blade modelization. When q is such a number, the state vector $\tilde{\boldsymbol{x}}(t) \in \Re^n$ has $n = 2q$ components, and the model can be made more accurate by increasing the number of modes. The dependence of the system matrices on velocity can be captured as follows. Let $\mu(t)$ denote the rotor advance ratio, namely, the dimensionless quantity

$$\mu(t) = V(t)/\Omega R \tag{2}$$

where $V(t)$ is the aircraft velocity and R is the rotor radius. Typically, this quantity takes values from 0 (hover) to 0.3 (high velocity). The approach described here can be extended to a wider range of velocities with a more accurate modelization of the rotor blade. For simplicity, in the sequel the dependence of μ on time is often omitted.

Because of the modeling assumptions, the system matrices $A(t)$, $B(t)$, $C(t)$, and $D(t)$ have the following structure, as discussed in Ref. 16:

$$A(t) = A_0(\mu) + A_{1c}(\mu)\cos(\Omega t) + A_{1s}(\mu)\sin(\Omega t)$$
$$B(t) = B_0(\mu) + B_{1s}(\mu)\sin(\Omega t) + B_{2c}(\mu)\cos(2\Omega t)$$
$$C(t) = C_0(\mu) + C_{1c}(\mu)\cos(\Omega t) + C_{1s}(\mu)\sin(\Omega t) + C_{2c}(\mu)\cos(2\Omega t)$$
$$D(t) = D_0(\mu) + D_{1s}(\mu)\sin(\Omega t) + D_{2c}(\mu)\cos(2\Omega t) \tag{3}$$

As one can note from Eqs. (3), the system matrices depend on t both directly and indirectly, via the rotor advance ratio μ. For the dependence of matrices $A_0(\mu)$, $A_{jc}(\mu)$, and $A_{js}(\mu)$, $j = 1, 2$, on μ, it can be shown that these matrices can be expressed as a power series limited to the second power[27]:

$$A_0(\mu) = A_{00} + A_{01}\mu + A_{02}\mu^2, \qquad A_{jc}(\mu) = A_{j1c}\mu + A_{j2c}\mu^2$$
$$A_{js}(\mu) = A_{j1s}\mu + A_{j2s}\mu^2 \tag{4}$$

Analogous expressions hold for $B(t)$, $C(t)$, and $D(t)$. From Eqs. (3) and (4), it is apparent that in hover (helicopter standing, $\mu = 0$) the system matrices $A(t)$, $B(t)$, $C(t)$, and $D(t)$ are constant, whereas in forward flight ($\mu = \text{const}$), they are periodic with period $T := 2\pi/\Omega$: $A(t) = A(t + T)$ and analogously for the remaining matrices. If one moves to considering a generic flight schedule, the periodic behavior of matrices is lost, and a more complex variability has to be taken into account.

Note that, whereas the functional variability of the system matrices depends on the flight schedule of the helicopter, the vibration disturbance $\tilde{d}(t)$ is always periodic because it is the effect of the rotation of the blades (which takes place at constant velocity). Therefore, independently of the flight conditions, the main angular frequency of the disturbance $\tilde{d}(t)$ is $N\Omega$. The numerical data used in the simulations refer to the helicopter Agusta A109 for which $\Omega = 40.32$ rad/s and $R = 5.5$ m: This helicopter has a four-bladed and fully articulated rotor.

Remark (system matrices): For the subsequent developments, it is useful to write the system matrices as

$$A(t) = A_0(t) + \sum_{j=1}^{2}[A_{jc}(\mu)\cos(j\Omega t) + A_{js}(\mu)\sin(j\Omega t)] \tag{5}$$

and analogously for $B(t)$, $C(t)$, and $D(t)$. Obviously, Eq. (5) is equivalent to expression (3) by adding some null matrices. Now, when Eqs. (4) is substituted here, one can obtain an expression of $A(t)$ in terms of the components of the following vector (with 15 elements):

$$\delta(t) = [1, \mu, \mu^2, \sin(\Omega t), \mu\sin(\Omega t), \mu^2\sin(\Omega t), \cos(\Omega t), \ldots, \mu^2\cos(2\Omega t)] \tag{6}$$

Precisely, denoting with $\delta_i(t)$, $i = 0, 1, 2, \ldots, 14$, the components of the vector (6), it follows that

$$A(t) = \delta_0(t)A_{0,0} + \delta_1(t)A_{0,1} + \delta_2(t)A_{0,2} + \cdots + \delta_{14}(t)A_{2,2c} \tag{7}$$

Analogous expressions hold true for $B(t)$, $C(t)$, and $D(t)$.

B. Active Control Problem

The purpose of the active control system is to superimpose to the pitch command of the pilot a variation to reject the periodic component of the shear force while keeping the intensity of the pitch angle variation limited enough to avoid interference with the pilot decisions. As a matter of fact, the linearity of the adopted model is a consequence of only small perturbations in the pitch angle being allowed. In this regard, note that the signal $u(t)$ in the model is the variation of the pitch angle with respect to the pitch angle decided by the pilot. The control problem is formulated as follows: Reduce the effect of $\tilde{d}(t)$ on $y(t)$ by acting on $u(t)$. For the design techniques presented in the sequel, it is useful to translate the disturbance from the output to the input, to replace system (1) with the equivalent system:

$$\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t) + B(t)[u(t) + d(t)] + v(t)$$
$$\boldsymbol{y}(t) = C(t)\boldsymbol{x}(t) + D(t)[u(t) + d(t)] + w(t) \tag{8}$$

Such a translation is possible thanks to the linearity of the system. However, two important issues should be underlined at this point.

1) The state vector $\tilde{\boldsymbol{x}}(\cdot)$ of system (1) is different from the state vector $\boldsymbol{x}(\cdot)$ of system (8) because of the disturbance translation. However, the two models are input/output equivalent, in the sense that the effect of the input equivalent disturbance (IED) $d(\cdot)$ on $\boldsymbol{y}(\cdot)$ in system (8) is the same as the effect of $\tilde{d}(\cdot)$ on $\boldsymbol{y}(\cdot)$ in system (1).

2) The spectral characteristics of $d(\cdot)$ may be quite different from those of $\tilde{d}(\cdot)$, the only exception being the hovering condition. Indeed, in hover, the system is time invariant and the spectral lines of the IED overlap with those of $\tilde{d}(\cdot)$ (with different intensities and phases). In forward flight, however, the system is time periodic, so that, even though the output disturbance reduces to a finite number of frequencies, infinitely many harmonics are required at the input to replicate its effect. In particular, although $\tilde{d}(\cdot)$ is periodic with angular frequency $N\Omega$, the input equivalent disturbance $d(\cdot)$ must be periodic with angular frequency Ω (also see Ref. 28). Thus, there is a frequency spawn effect because the spectral content of $d(\cdot)$ covers a much wider range of frequencies than that of $\tilde{d}(\cdot)$.

III. LFR of the Rotor Blade Model

At the basis of the development of our control system, there is the observation that the helicopter rotor blade model (1) can be given an LFR. Such a representation describes a system as a feedback structure whose forward part is purely time invariant, whereas the backward part contains the whole time variability of the model. For systems with such a structure, powerful control techniques are available (e.g., see Ref. 25).

In our case, we observe that the original system (1), deprived of the disturbance $\tilde{d}(t)$, can be rewritten in the form

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ y(t) \end{bmatrix} = M(t) \begin{bmatrix} \tilde{x}(t) \\ u(t) \\ v(t) \\ w(t) \end{bmatrix} \tag{9}$$

where

$$M(t) = \begin{bmatrix} A(t) & B(t) & I & 0 \\ C(t) & D(t) & 0 & I \end{bmatrix} \tag{10}$$

In turn, recall that all of the rotor model matrices present an affine structure, for example, the matrix $A(t)$ has the form (7), then the matrix $M(t)$ can be rewritten in the following way:

$$M(t) = \delta_0(t)M_{0,0} + \delta_1(t)M_{0,1} + \delta_2(t)M_{0,2} + \cdots + \delta_{14}(t)M_{2,2c} \tag{11}$$

In view of the particular structure of this model, we can easily find an LFR of model (9–11). More precisely, letting

$$\tilde{\delta}_1(t) = \mu, \qquad \tilde{\delta}_2(t) = \cos(\Omega t), \qquad \tilde{\delta}_3(t) = \sin(\Omega t)$$
$$\tilde{\delta}_4(t) = \cos(2\Omega t), \qquad \tilde{\delta}_5(t) = \sin(2\Omega t) \tag{12}$$

it is possible to find three constant matrices U, V, and S of suitable dimensions and five positive integers r_i, with $i = 1, \ldots, 5$, such that the matrix $M(t)$ can be rewritten as follows:

$$M(t) = M_{0,0} + U\Delta(t)[I - S\Delta(t)]^{-1}V \tag{13}$$

where

$$\Delta(t) = \text{diag}(\tilde{\delta}_1 I_{r1}, \tilde{\delta}_2 I_{r2}, \tilde{\delta}_3 I_{r3}, \tilde{\delta}_4 I_{r4}, \tilde{\delta}_5 I_{r5}) \tag{14}$$

Expression (13) can be found via singular value decomposition (SVD) techniques by exploiting the rational dependence of matrix $M(t)$ on functions $\tilde{\delta}_i(t)$. This corresponds to a standard procedure[25] that has been briefly outlined in Appendix A. According to a common practice, one can also write

$$M(t) = \mathcal{F}\left(\begin{bmatrix} M_{0,0} & U \\ V & S \end{bmatrix}, \Delta(t)\right) \tag{15}$$

This type of system description is familiar in the realm of robust control: Following the ideas in Ref. 26 and others, we have achieved the important goal of confining the time variability of the original rotor model (1) in the return block of a feedback system, whose forward part is purely time invariant (Fig. 1).

IV. Control Rationale

In Ref. 5, the disturbance attenuation problem has been faced in the forward flight case only, resorting to optimal control techniques. More precisely, the control rationale used there calls for two loops: The inner loop is used to reconstruct the IED to compensate the disturbance $\tilde{d}(t)$ acting on the output of model (1); the outer loop has the task of stabilizing (by means of an LQ approach) the whole system. In this paper, we design a control system for a generic flight schedule according to the following steps (Fig. 2).

1) An effective estimate of the IED is obtained by means of the internal model control (IMC) principle[29]: Under the assumption that an effective stabilizing control signal $u_s(t)$ for the helicopter model (9) [see also Eq. (1)] is available for any flight condition, a sensible estimate $\hat{\tilde{d}}(\cdot)$ of the periodic disturbance $\tilde{d}(\cdot)$ can be obtained by subtracting the measured output $\boldsymbol{y}(t)$ to the output of an suitably complex internal model of the rotor blade (9) (Fig. 2). Moreover, from the estimate of the disturbance $\tilde{d}(\cdot)$, the IED signal can be recovered by means of filtering techniques and periodic frequency domain concepts that will be described in Sec. IV.A. This rationale is represented by the inner loop of Fig. 2.

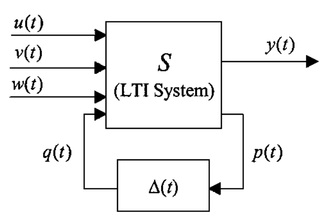


Fig. 1 LFR of the rotor blade model.

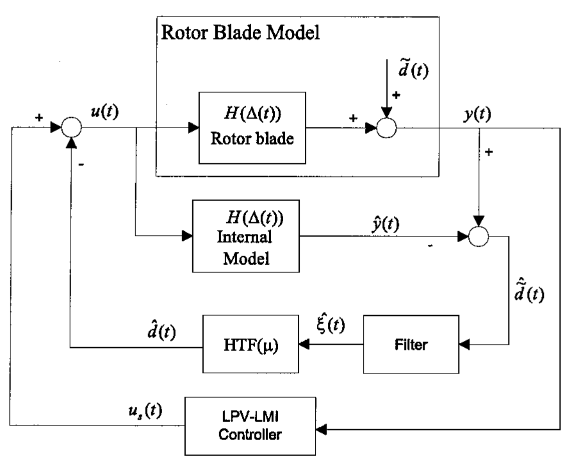


Fig. 2 Control rationale.

2) The outer loop computes the stabilizing control signal $u_s(t)$ by means of an LPV control law guaranteeing robust stability independently of the flight condition. This will be the subject of Sec. IV.B.

A. Reconstruction of the IED

In the vibration attenuation problem, a major design task is the selection of the frequencies to be attenuated. The problem is often limited to that of canceling the first harmonic (at angular frequency $N\Omega$) in the output disturbance. However, in general, one may consider a number of spectral lines, for example, $N\Omega, 2N\Omega, \ldots, mN\Omega$, to be canceled. The output disturbance can be expanded in Fourier series as

$$\tilde{d}(t) = \sum_k \tilde{d}^{(k)}(t) \tag{16}$$

and the appropriate state-space model for a generic harmonic at frequency $kN\Omega$ is given by the second-order model:

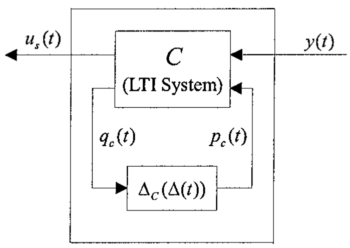


Fig. 3 Structure of the LPV stabilizing controller.

$$\dot{\xi}^{(kN)}(t) = W^{(kN)}\xi^{(kN)}(t) + z^{(kN)}(t), \qquad \tilde{d}^{(kN)}(t) = h\xi^{(kN)}(t) \tag{17}$$

where

$$W^{(k)} = \begin{bmatrix} 0 & -(k\Omega)^2 \\ 1 & 0 \end{bmatrix}, \qquad h = [1 \quad 0] \tag{18}$$

Consequently, by piling up all state vectors of the considered harmonics $\xi^{(N)}(t), \xi^{(2N)}(t), \ldots, \xi^{(mN)}(t)$, one obtains the overall model for the output disturbance as follows:

$$\dot{\xi}(t) = W\xi(t) + z(t), \qquad \tilde{d}(t) = H\xi(t) \tag{19}$$

with

$$W = \begin{bmatrix} \boxed{W^{(N)}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \cdots & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{W^{(2N)}} & \cdots & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \vdots\ \vdots & \vdots\ \vdots & \ddots & \vdots\ \vdots \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \cdots & \boxed{W^{(mN)}} \end{bmatrix}$$

$$H = \begin{bmatrix} \boxed{h} & \boxed{h} & \cdots & \boxed{h} \end{bmatrix}$$

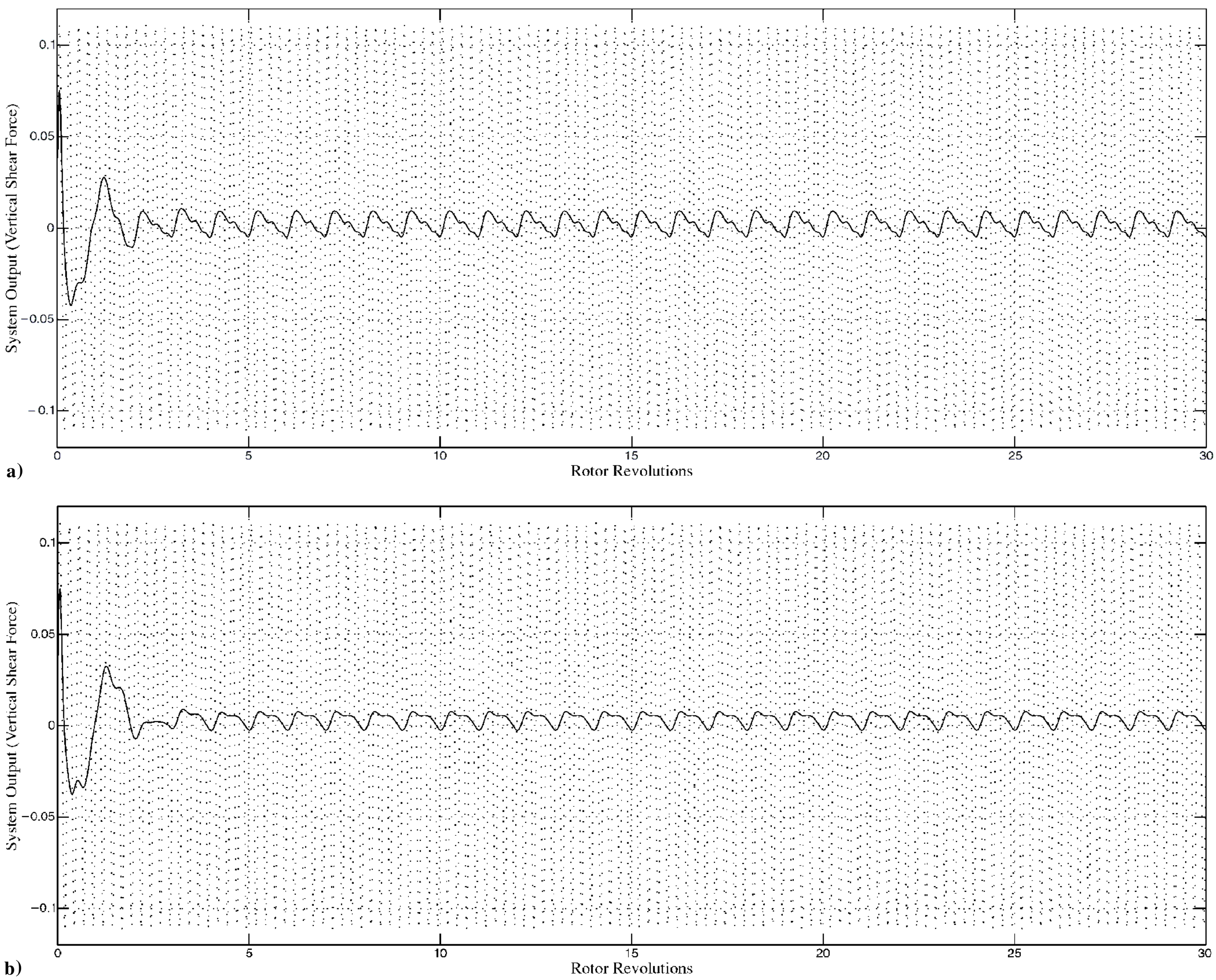


Fig. 4 Forward flight: $\mu = 0.3$—vertical shear force at the root of the blade without (· · · ·) and with active vibration control (——). The experiments correspond to the cases: a) $k_2 = 1.1$ and b) $k_2 = 1.15$.

A first step toward the reconstruction of the IED is the estimation of the state $\xi(t)$ of the disturbance model (19). From the estimate $\hat{\tilde{d}}(\cdot)$ of $\tilde{d}(\cdot)$ obtained by means of the IMC principle, one can reconstruct the state $\xi(\cdot)$ of the disturbance model (19). Because the output disturbance model (19) is linear and time invariant, the filter considered to this purpose is linear and time invariant as well:

$$\dot{\boldsymbol{x}}_f(t) = A_f \boldsymbol{x}_f(t) + B_f \hat{\tilde{d}}(t), \qquad \hat{\xi}(t) = C_f \boldsymbol{x}_f(t) + D_f \hat{\tilde{d}}(t) \quad (20)$$

As filtering criterion, an H_∞ rationale was adopted[18,30,31] by imposing a limited effect of the noise $z(t)$ on the estimation error $\hat{\xi}(t) - \xi(t)$ according to the condition

$$\sup_{z \in L_2 - \{0\}} \frac{\|\hat{\xi} - \xi\|_{L_2}}{\|z\|_{L_2}} < \gamma_f \quad (21)$$

When the estimate $\hat{\xi}(t)$ of the state of model (19) is obtained, it is possible to derive the estimate of the kth harmonic $\tilde{d}^{(kN)}(t)$ of the output disturbance. Indeed, if we write

$$\tilde{d}^{(kN)}(t) = \alpha^{(kN)} \sin(kN\Omega t) + \beta^{(kN)} \cos(kN\Omega t) \quad (22)$$

$\alpha^{(kN)}$ and $\beta^{(kN)}$ must satisfy the equations

$$\xi_1^{(kN)}(t) = \alpha^{(kN)} \sin(kN\Omega t) + \beta^{(kN)} \cos(kN\Omega t)$$
$$\xi_2^{(kN)}(t) = -\left[\alpha^{(kN)}/kN\Omega\right] \cos(kN\Omega t) + \left[\beta^{(kN)}/kN\Omega\right] \sin(kN\Omega t) \quad (23)$$

where $\xi_1^{(kN)}(t)$ and $\xi_2^{(kN)}(t)$ are the components of $\xi^{(kN)}(t)$. Parameters $\alpha^{(kN)}$ and $\beta^{(kN)}$ can be obtained by solving this system with the estimated state in place of the true one.

With knowledge of $\alpha^{(kN)}$ and $\beta^{(kN)}$, the computation of the harmonics of the IED can be performed via the harmonic transfer function (HTF). This method, concisely outlined in Appendix B, puts into correspondence the input and output harmonics of a periodic system operating in a periodic regime. Note that, to a single harmonic at the output, there corresponds an input with infinitely many harmonics. By the repeating of the single harmonic procedure for $k = 1, 2, \ldots, m$, the appropriate $d(\cdot)$ can be found by superposition of the various contributions needed for the single harmonics at the output.

Remark (HTF): As a tool to invert a periodic system operating in a periodic regime, the HTF is rigorously valid in forward flight only. However, our simulation experience shows that one can use it even in a generic flight condition with satisfactory results. The reason is probably that, considering realistic variations rates of the velocity, the passage from a periodic regime to another periodic regime takes place smoothly enough. Obviously, to cope with a general flight condition, we consider an HTF tuned according to the (measured) advance ratio μ (Fig. 2).

B. Synthesis of the Control Law

In this section we face the problem of finding a sensible stabilizing control law for the rotor blade model (9). The same problem was addressed in Ref. 5 by resorting to the separation principle in

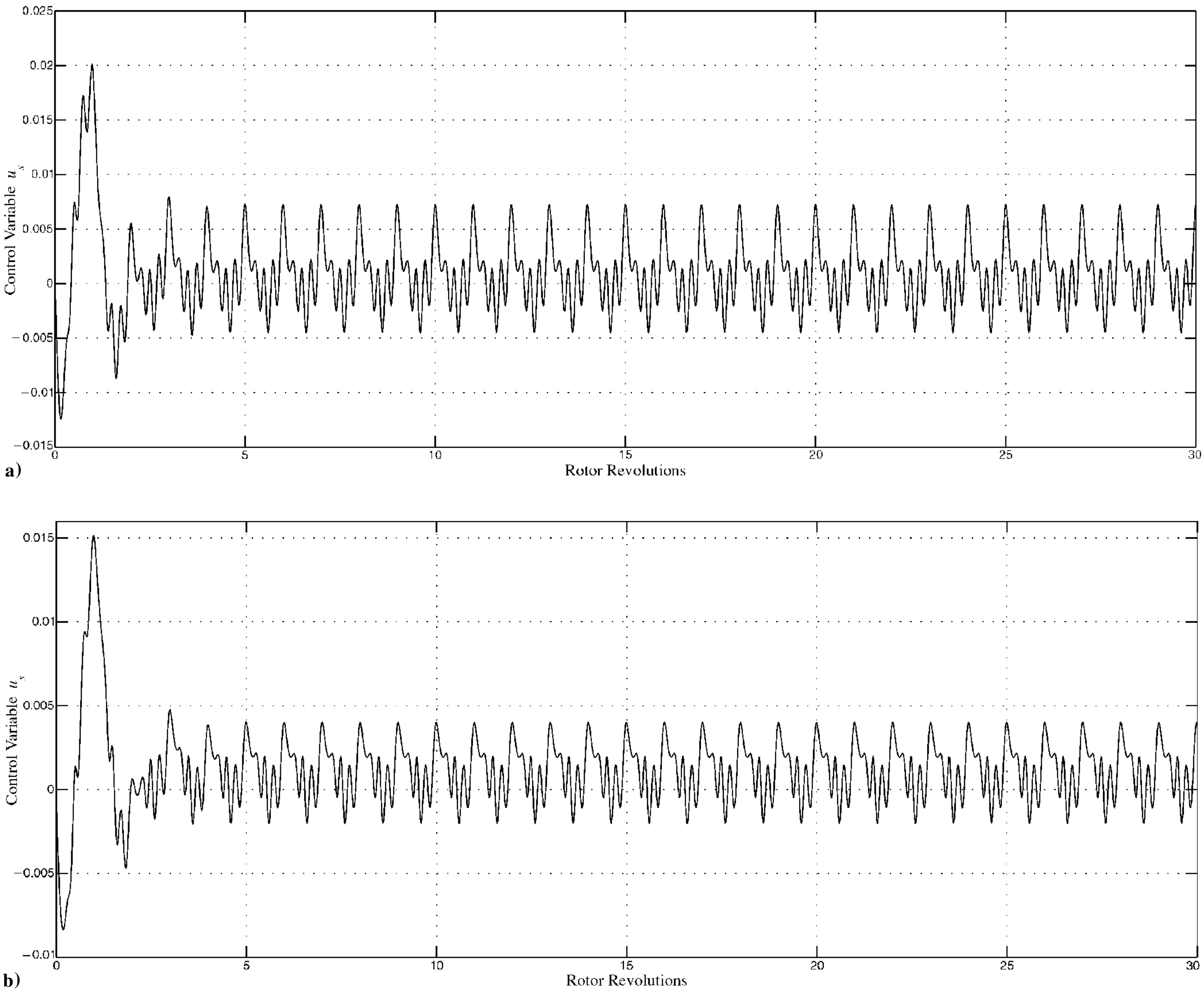


Fig. 5 Forward flight: μ = 0.3—control variable u_s (in radians). The experiments correspond to the cases: a) k_2 = 1.1 and b) k_2 = 1.15.

an LQG framework for periodic systems. Here, we propose an H_∞ robust control approach that does not exploit any separation principle. Precisely, we observe that in our problem, the advance ratio μ is measured. Consequently, we can resort to LPV ideas[19,32–35] to work out a stabilizing controller valid for any flight condition. The implemented LPV controller (Fig. 3) consists of a linear time invariant system (denoted by C) and a feedback block $[\Delta_c(\cdot)]$. The $\Delta_c(\cdot)$ law is deduced from function $\Delta(\cdot)$, which in turn is computed from the advance ratio μ [see Eqs. (12) and (14)]. In this manner, we can face the vibration control problem of the rotor in general terms disregarding the specific flight condition of the aircraft: This is the most significant advantage of the control technique described here with respect to the periodic strategy described in Refs. 5 and 17. In fact, as pointed out in our previous works, a periodic controller calculated for a certain velocity can exhibit poor performances in a different flight condition. Therefore, to face the generic flight condition by means of a periodic strategy, one should schedule both the control and the filtering laws according to the actual advance ratio: as previously pointed out,[17] such an approach requires a consistent amount of periodic controllers and filters to be scheduled.

The LPV stabilizing control law proposed here has the following state-space structure:

$$\dot{\boldsymbol{x}}_c(t) = A_c\boldsymbol{x}_c(t) + B_c\begin{bmatrix}\boldsymbol{y}(t)\\ p_c(t)\end{bmatrix} \tag{24}$$

$$\begin{bmatrix}u_s(t)\\ q_c(t)\end{bmatrix} = C_c\boldsymbol{x}_c(t) + D_c\begin{bmatrix}\boldsymbol{y}(t)\\ p_c(t)\end{bmatrix} \tag{25}$$

$$p_c(t) = \Delta_c[\Delta(t)]q_c(t) \tag{26}$$

where A_c, B_c, C_c, and D_c define the linear time invariant part C of the controller represented in Fig. 3. For the specific design of matrices A_c, B_c, C_c, and D_c and function $\Delta_c(\cdot)$, one can resort to different criteria: In our case we have adopted the LPV–H_∞ approach presented in Ref. 19. This technique can be applied to plants for which an LFR is available. It has the distinctive feature that the synthesis approach does not require any knowledge on the rate of variation of the scheduling parameters contained in the backward part of the rotor blade LFR model (see Sec. III), that is, the rate of variation of functions $\tilde{\delta}_i(t)$ [see Eqs. (12–14)]. Only the knowledge of the ranges of variation of functions $\tilde{\delta}_i(t)$ is necessary. In fact, to apply the theory of Ref. 19, one has to make a convexity assumption on the set where the scheduling parameters $\tilde{\delta}_i(t)$ range

$$\forall i \in [1, 5], \qquad \tilde{\delta}_i^m \le \tilde{\delta}_i \le \tilde{\delta}_i^M \tag{27}$$

Now, it is fairly apparent that the parameters $\tilde{\delta}_2$, $\tilde{\delta}_3$, $\tilde{\delta}_4$, and $\tilde{\delta}_5$, being sinusoidal functions, trivially satisfy Eq. (27). The scheduling parameter $\tilde{\delta}_1$ is the advance ratio μ, and it is obviously limited for

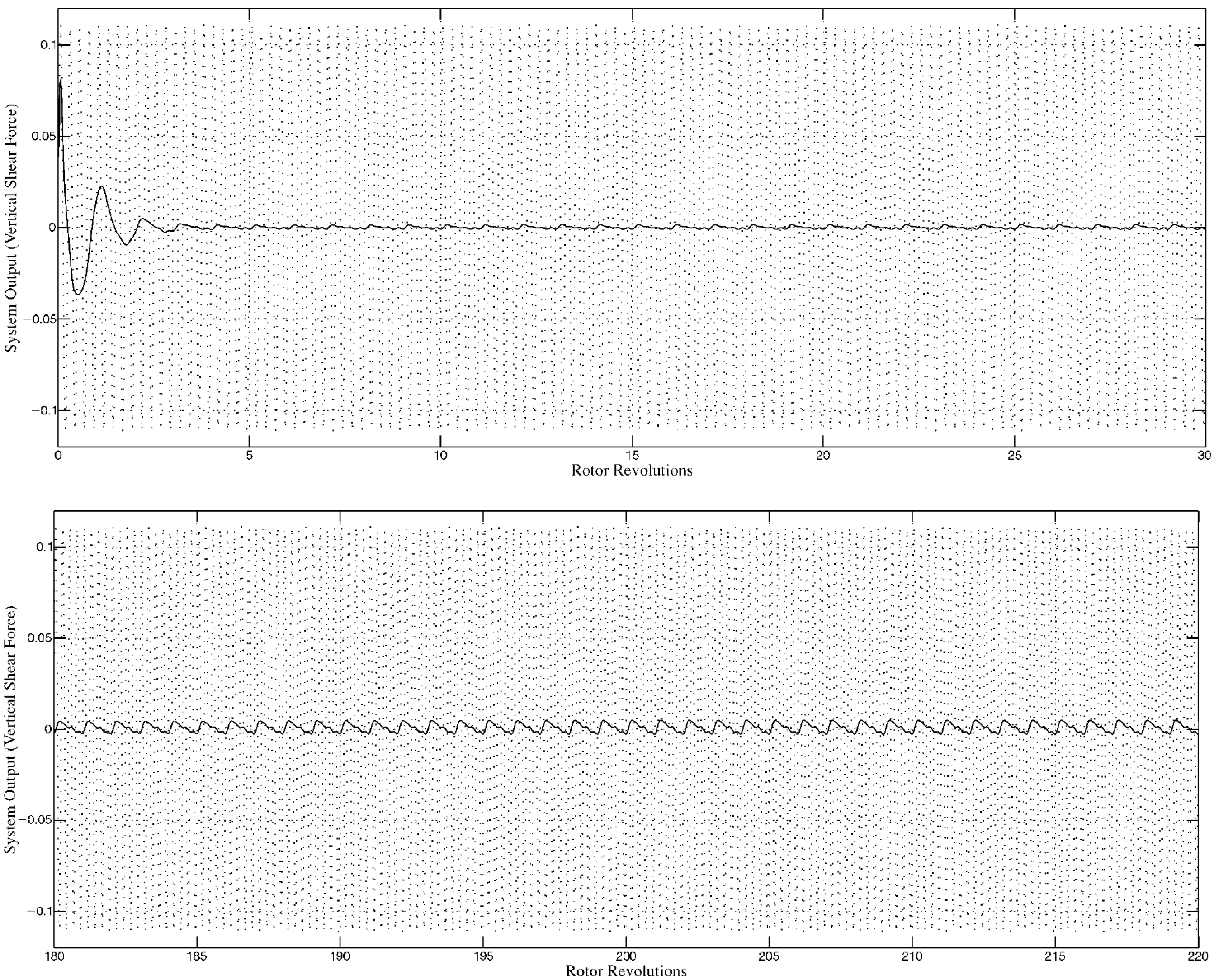


Fig. 6 Flight at variable velocity: μ varying as in Fig. 10—vertical shear force at the root of the blade without (· · · · ·) and with active vibration control (——). The experiment corresponds to the cases k_2 = 1.1.

technological reasons: $0 \leq \tilde{\delta}_1 \leq \mu_{\max}$. As a reasonable value one can take $\mu_{\max} = 0.3$. See Ref. 19 for the details on the synthesis problem and to Refs. 36 and 37 for more details on the theoretical aspects.

Remark (LPV synthesis problem): The LPV synthesis problem associated with a continuous-time and time-varying system calls for an infinite-dimensional set of LMIs. To overcome this difficulty, many approaches have been proposed.[32] We have adopted the synthesis procedure proposed in Ref. 19. This approach is based on the full block S-procedure[36,38] and (in a conservative way) on a single constant Lyapunov function for all feasible values of the time-varying parameters. That the set of all feasible $\Delta(t)$ is contained in a convex hull Δ [see Eq. (27)] sets the basis for the use of the control design technique proposed in Ref. 19. Precisely, let $\Delta_1, \Delta_2, \ldots, \Delta_K$ be a minimal set of generators of the convex hull Δ. In our case, $K = 2^5 = 32$ because as pointed out by Eq. (14), $\Delta(t)$ is defined on the basis of five functions $\tilde{\delta}_i(t)$. The synthesis technique described in Ref. 19 reduces an infinite-dimensional LMI problem to a finite number of LMIs by imposing some LMI constraints on a suitable full-block scaling matrix for each generator Δ_i of the convex hull. More precisely, letting Q be this scaling matrix of suitable dimensions, by applying the full block S-procedure we obtain the following K LMI constraints:

$$\forall i = 1, 2, \ldots, K, \qquad \begin{bmatrix} \Delta_i \\ I \end{bmatrix}^T Q \begin{bmatrix} \Delta_i \\ I \end{bmatrix} > 0 \tag{28}$$

Further details about this technique can be found in Refs. 19 and 36. □

The control design we have performed is based on the H_∞ performance requirement

$$\sup_{w_e \in L_2 - \{0\}} \frac{\|z_c\|_{L_2}}{\|w_e\|_{L_2}} < \gamma_c \tag{29}$$

where $z_c(\cdot)$ is a suitable performance signal, $w_e(\cdot) := [v(\cdot)^T \,|\, w(\cdot)^T]^T$ is the disturbance acting on the whole system (9), and γ_c is a performance index to be tuned. In our problem, an effective choice of $z_c(\cdot)$ is given by

$$z_c(t) = k_1 \begin{bmatrix} y(t) \\ k_2 u_s(t) \end{bmatrix} \tag{30}$$

where k_1 and k_2 are weighting coefficients to be chosen by the designer. As it will be pointed out in the validation experiments reported in the following section, the usefulness of the performance coefficient k_2 appearing in the expression of $z_c(\cdot)$ consists of the possibility of limiting the amplitude of the control signal $u_s(\cdot)$ generated by the stabilizing LPV controller and, hence, in the possibility of avoiding an excessive interference with the pilot decisions. The role of the coefficient k_1 is purely numeric because it allows the numerical stability of the synthesis procedure to improve.

V. Validation Experiments

In this section, we propose some numerical experiments to show how the control rationale of Fig. 2 allows the disturbance rejection

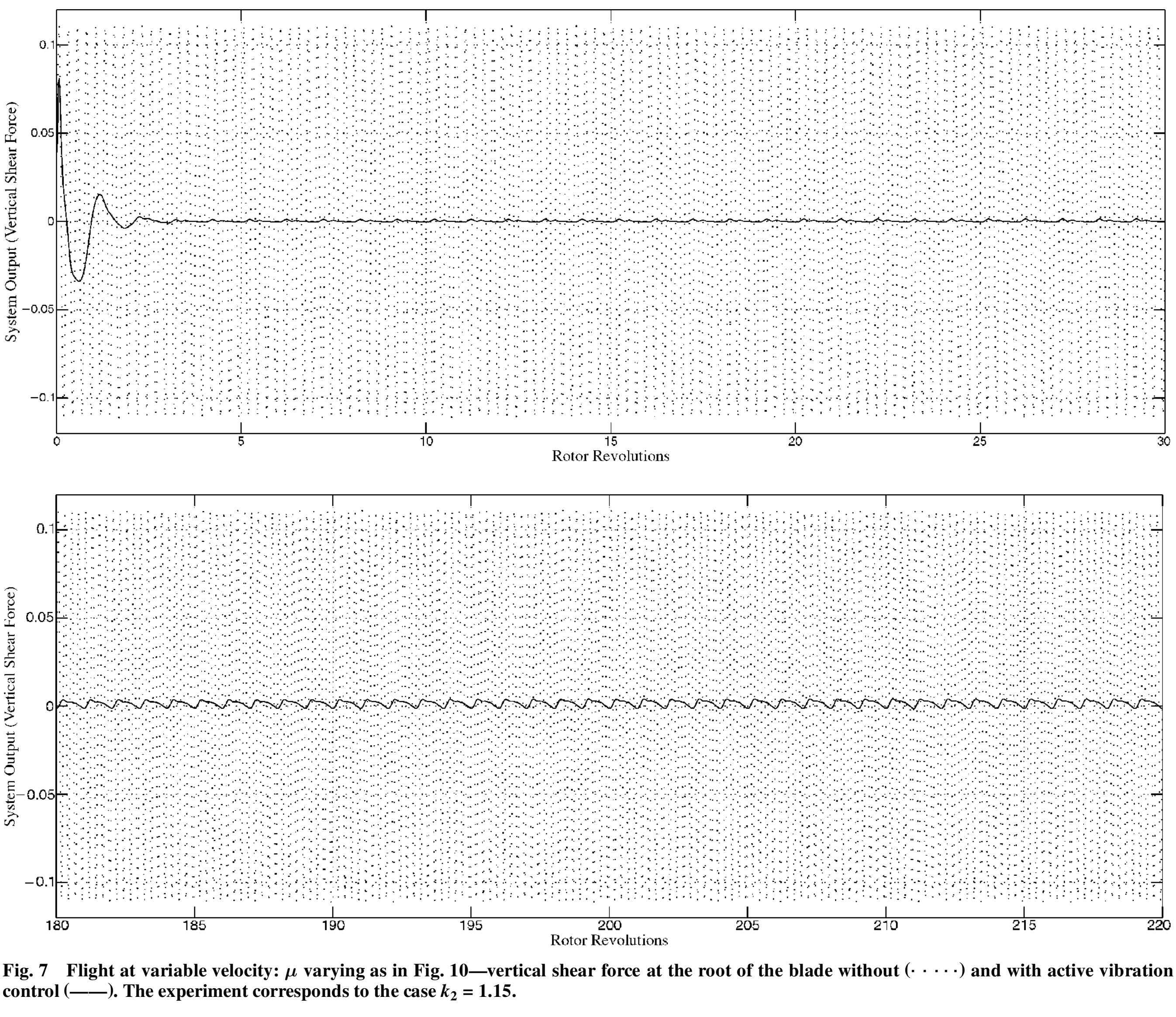


Fig. 7 Flight at variable velocity: μ varying as in Fig. 10—vertical shear force at the root of the blade without (· · · · ·) and with active vibration control (——). The experiment corresponds to the case k_2 = 1.15.

to perform in a versatile way. Of course, this requires the preliminary tuning of the performance coefficients k_1, k_2, γ_f, and γ_c [see Eqs. (21), (29), and (30)].

According to an extensive set of simulation trials, it turns out that a good performance can be achieved by taking as performance coefficients γ_f, γ_c, and k_1 the values reported in Table 1. As noted before, the coefficient k_2 can be used as a tuning knob to regulate the amplitude of the control variable $u_s(\cdot)$ to avoid an excessive interference with the pilot decisions.

A first set of simulation experiments have been performed with reference to the forward flight condition associated with $\mu = 0.3$. The diagrams of the output and of the control input u_s are presented in Figs. 4 and 5 respectively. We have considered the following values of the coefficient $k_2 = \{0.1, 0.15\}$.

A second family of experiments refer to a ramp variation of the advance ratio from 0.15 to 0.25 followed by a period in which the advance ratio is kept constant to the value of 0.25 (see Figs. 6–10). For simplicity, we will consider a constant amplitude for the disturbance signal d(t) for all possible values of the advance ratio. Clearly, our control rationale remains valid even when d(t) varies in amplitude with the advance ratio. This second set of simulations refers to experiments lasting 250 revolutions: Because of the very long duration of these experiments (which is compliant with a realistic acceleration of an helicopter), in Figs. 6–10 we plotted the data corresponding to the first 30 revolutions and the data corresponding to the rotor revolutions from 180 to 220 (note that from revolution 200 onward, the advance ratio is kept constant).

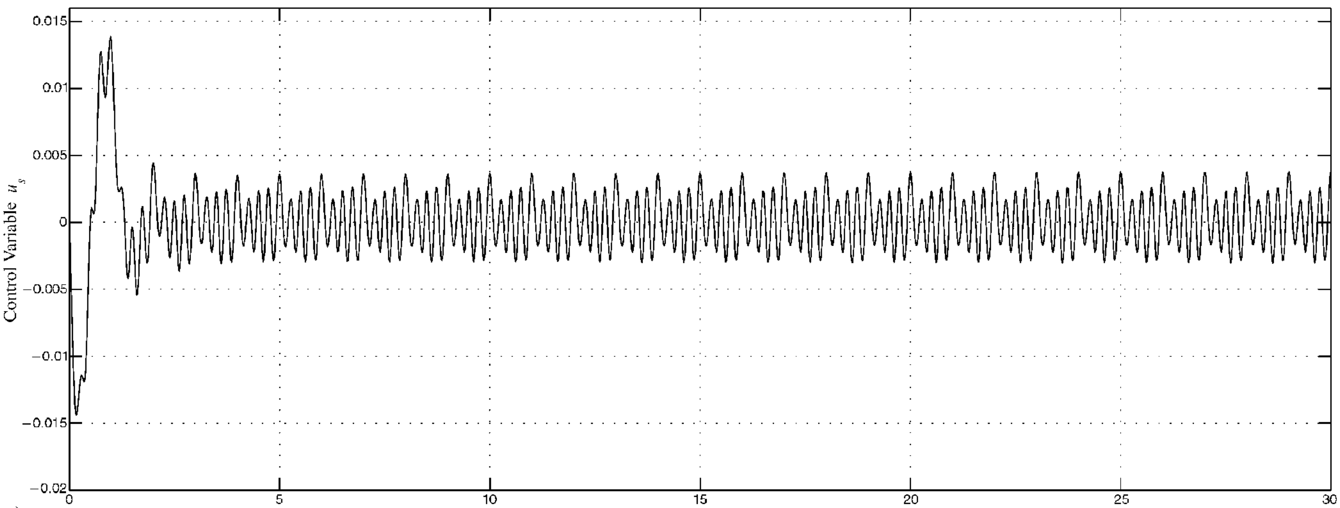
Table 1 Performance coefficients

Coefficient	Value
γ_f	0.1
γ_c	0.05
k_1	0.1

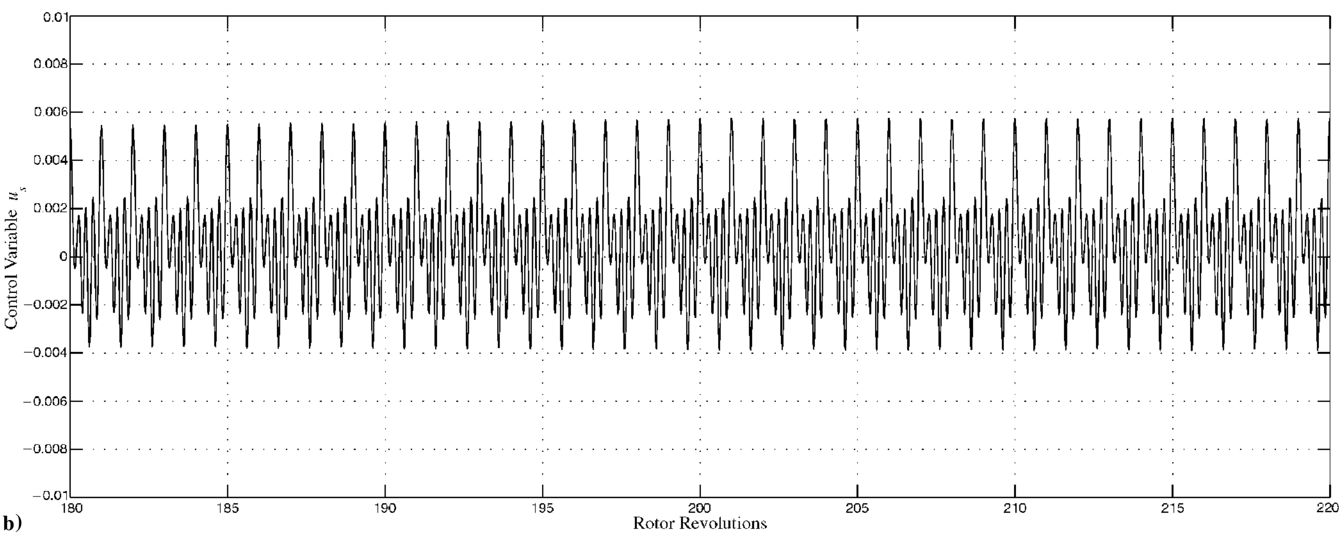
Also in this case, we consider different experiments associated with $k_2 = \{0.1, 0.15\}$. The results of the various simulations clearly show that the intensity of the control variable is dependent on the value assumed by k_2: Figures 6 and 8 refer to the case $k_2 = 0.1$, whereas Figs. 7 and 9 refer to the case $k_2 = 0.15$. Moreover, Figs. 8 and 9 show that the control problem becomes more demanding for high values of the advance ratio.

Notice that the control input signal exhibits a diverging pattern over the first 200 revolutions (compare Figs. 8a and 9a, where the first 30 rotor revolutions are plotted, with the Figs. 8b and 9b, where the rotor revolutions from 180 to 220 are plotted). This is obviously due to the progressive increase in the value of μ. When μ becomes constant (from the revolution 200 onward) the oscillation in the signal $u_s(t)$ maintains a constant amplitude (see Figs. 8b and 9b). Finally, from Figs. 8 and 9, it appears that the maximum amplitude of the control signal looks compatible with the model linearity assumption. Moreover, with a suitable choice of k_2, the amplitude of the control signal $u_s(t)$ can be kept small enough to avoid interference with the pilot decisions.

To better evaluate the effectiveness of the synthesized controllers in terms of the tradeoff between disturbance rejection effect and



a)



b)

Fig. 8 Flight at variable velocity: μ varying as in Fig. 10—control variable u_s (in radians). The experiment corresponds to the case k_2 = 1.1.

energy of the control input $u_s(\cdot)$, the following cost functions are introduced:

$$J_1(k_2) = \frac{1}{t_f - t_i}\int_{t_i}^{t_f} y(t)^2\,\mathrm{d}t\Big|_{\text{closed loop}}$$

$$J_2(k_2) = \frac{1}{t_f - t_i}\int_{t_i}^{t_f} u_s(t)^2\,\mathrm{d}t, \qquad J_3(k_2) = \frac{J_1(k_2)}{J_2(k_2)} \quad (31)$$

Roughly, $J_1(\cdot)$ is the output signal energy in closed loop, whereas $J_2(\cdot)$ is an indication of the corresponding control effort. Table 2 shows that as k_2 increases the closed-loop output energy decreases so that a better disturbance rejection is achieved. Correspondingly, the energy of $u_s(\cdot)$ decreases as well, but the quotient $J_3(\cdot)$ is increasing. These considerations can be used for a sensible selection of k_2.

Remark (IBC approach): The IBC approach may be implemented without necessarily using the swashplate as actuator device. This is why, as pointed out by many authors (e.g., see Ref. 9), this technique allows control of each blade individually without restriction on the applied frequencies. When the IBC is implemented by means of the swashplate, there is an obvious limitation that is related to the band of the actuator. In this respect, notice that our approach allows selection of the number of spectral lines in the control variable by a sensible truncation in the HTF calculation. For instance, in our reported simulations, we have considered a vibration disturbance consisting of a single harmonic (at 4Ω), and we have allowed in the control signal all frequencies $k\Omega$ with k ranging from 1 to 11.

Table 2 Performance cost functions

Function	$k_2 = 1.1$	$k_2 = 1.15$	$k_2 = 1.2$
$J_1(k_2)$	3.3056×10^{-4}	3.0974×10^{-4}	2.7183×10^{-4}
$J_2(k_2)$	1.7849×10^{-4}	1.0989×10^{-4}	5.6939×10^{-5}
$J_3(k_2)$	1.8520	2.8186	4.7759

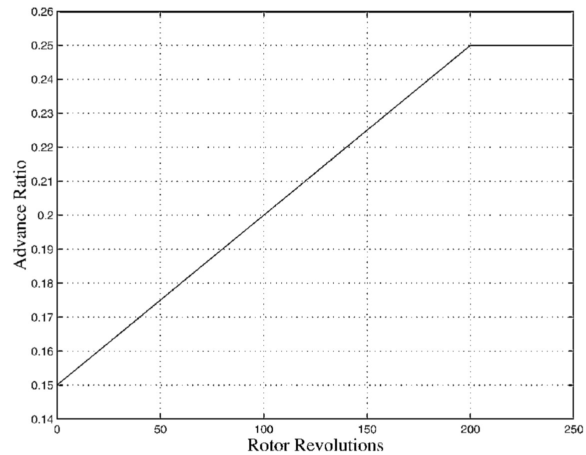


Fig. 10 Flight at variable velocity: μ varies from 0.15 to 0.25 in 200 rotor revolutions, then it is kept constant.

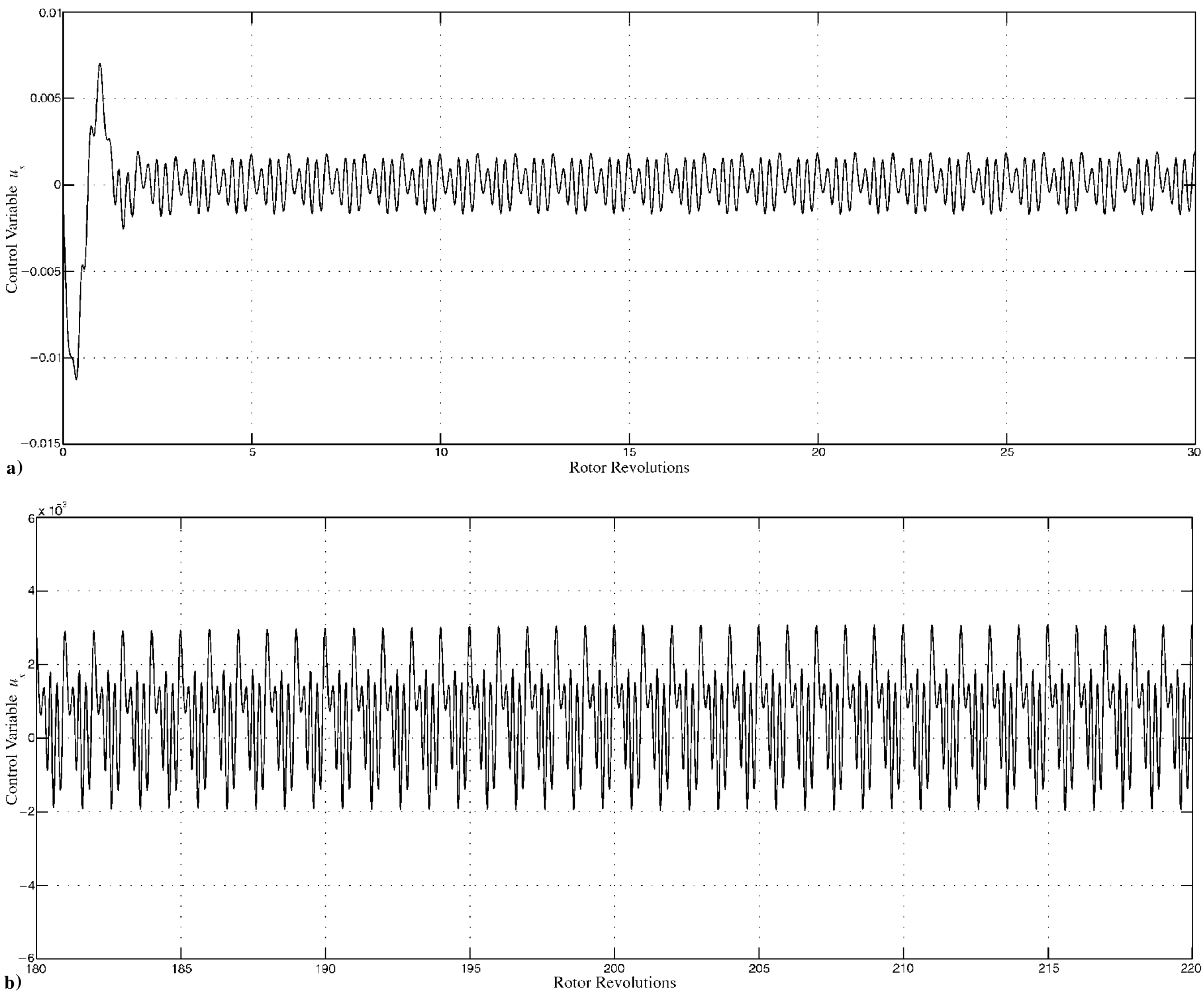


Fig. 9 Flight at variable velocity: μ varying as in Fig. 10—control variable u_s (in radians). The experiment corresponds to the case k_2 = 1.15.

VI. Conclusions

This paper addresses the problem of attenuating the vibrations induced in the helicopter fuselage by the main rotor within an IBC framework. This framework is characterized by the need of using time-varying models. We tackle the control problem for any flight condition by resorting to a LPV control strategy.

In the special case of level forward flight, the time variability takes a particular structure in that the system coefficients are periodic. Correspondingly, by exploiting periodic control techniques, a suitable controller can be designed for a specific velocity. Unfortunately, the periodic control approach shows poor performance when a controller designed for a certain velocity is used for another velocity. To overcome this problem, a gain scheduling of periodic controllers would be required. This may be impractical because the number of the control laws that have to be considered turns out to be high. On the contrary, the LPV approach leads to a unique control law that exploits the measurements of the advance ratio as scheduling parameter to adapt the control rationale to the specific flight condition.

Moreover, the adopted LPV control technique can deal with plants subject to time variabilities with any rate of variation. What really matters is the knowledge of the range of variation of the scheduling parameters. This is the reason why an LPV technique also can be extended to the case in which the plant contains nonlinear effects.

Last, the LPV technique allows to face the fundamental problem of keeping limited the amplitude of the control signal to avoid an excessive interference with the piloting commands.

Our simulation experiments show a satisfactory behavior of the control system even in demanding situations. They can be still improved by allowing more frequencies in the spectrum of the input equivalent disturbance.

A final remark concerns the fact that our control design methodology relies on an H_∞ performance assessment. This has a twofold advantage: First, it is not necessary to have a priori knowledge about the intensity of the disturbances representing noise effects and unmodeled dynamics. Second, there is the possibility of resorting easy-to-use design knob to to tune the controller.

Appendix A: LFR of the Rotor Blade Model

As pointed out in Eq. (10), the rotor blade model is represented by a matrix having the following affine structure in some time functions $\delta_i(t)$ with $i = 1, 2, \ldots, 14$:

$$M(t) = M_0 + \sum_{i=0}^{14} M_i \delta_i(t) \tag{A1}$$

A simple way to achieve an LFR is to factorize matrices M_i as $M_i = U_i V_i$, where the number of columns of U_i is given by the rank r_i of M_i. In practice, such a factorization is found by means of SVD techniques, so that it is possible to drop out the negligible singular values. In this way, the number of columns of U_i is in fact the numerical rank of M_i. In such a way the matrix $M(t)$ of Eq. (A1) can be rewritten as follows:

$$M(t) = M_0 + U\Delta(t)V \tag{A2}$$

where

$$U = [U_1, \ldots, U_{14}], \qquad V = [V_1', \ldots, V_{14}']' \tag{A3}$$

and

$$\Delta(t) = \mathrm{diag}_{i=1}^{14}\left[\delta_i(t) I_{r_i}\right] \tag{A4}$$

This way of proceeding presents a substantial drawback concerning the computational effort required to synthesize an LPV controller. In fact, as pointed out in the Remark of Sec. IV.B, the synthesis of an LPV controller implies at least the generation of 2^K LMI constraints [see Eq. (28)] where K is the cardinality of a minimal set of generators of the convex hull Δ is which $\Delta(t)$ is contained. In our problem, it would turn out $K = 14$ [see Eq. (A4)] because $\Delta(t)$ is depending on 14 independent function $\delta_i(t)$. Consequently, the number of LMI constraints to be considered would be huge that is, 16,384 constraints.

Fortunately, each function $\delta_i(t)$ can be expressed terms of five functions only, the $\tilde{\delta}_i(t)$ specified in Eq. (12); precisely, the components $\delta_i(t)$ of vector (6) can be written as

$$\delta_i(t) = \prod_{j=1}^{5} \tilde{\delta}_j(t)^{n_{i,j}} h_{(i,j)} \tag{A5}$$

where $n_{i,j}$ are suitable positive integers, and

$$h_{(i,j)} = \begin{cases} 1 & \text{if} \quad \tilde{\delta}_j(t) \text{ is a factor for } \delta_i(t) \\ 0 & \quad\quad \text{otherwise} \end{cases} \tag{A6}$$

Consequently, each term of Eq. (A1) can be written as

$$M_i \delta_i(t) = M_i \prod_{j=1}^{5} \tilde{\delta}_j(t)^{n_{i,j}} h_{(i,j)} \tag{A7}$$

which, by factorizing matrix M_i, can assume the following form:

$$M_i \delta_i(t) = \prod_{j=1}^{5} [M_{i,j} \tilde{\delta}_j(t)]^{n_{i,j}} h_{(i,j)} \tag{A8}$$

In conclusion, the problem of finding the LFR of $M(t)$ [Eq. (A1)] can be faced by finding the LFR of each single term $M_{i,j}\tilde{\delta}_j(t)$. Indeed, the LFR of Eq. (A8) can be computed by multiplication of the LFR of these terms. In turn, by summation of the LFR of Eq. (A8), one can achieve the LFR of Eq. (A1). In such a way we can obtain an LFR for which K, that is, the cardinality of a minimal set of generators of the convex hull Δ, is 5 only. Consequently the number of synthesis LMI constraints is drastically reduced that is, from 16,384 to 32.

For multiplication, consider the simple case when

$$\delta_i(t) = \tilde{\delta}_1(t)\tilde{\delta}_2(t) \tag{A9}$$

and with

$$M_i \delta_i(t) = M_{i1}\tilde{\delta}_1(t) M_{i2}\tilde{\delta}_2(t) \tag{A10}$$

The LFR of this product can be found with the following algorithm.

1) If r_{i1} is the rank of M_{i1}, find an r_{i1} decomposition of M_{i1} as $M_{i1} = U_{i1} V_{i1}$. Analogously, find an r_{i2} decomposition of M_{i2} as $M_{i2} = U_{i2} V_{i2}$, where r_{i2} is the rank of M_{i2}. In such a way we can write

$$M_{i1}\tilde{\delta}_1(t) = \mathcal{F}\left(\begin{bmatrix} 0 & U_{i1} \\ V_{i1} & 0 \end{bmatrix}, \tilde{\delta}_1(t) I_{r_{i1}}\right)$$

$$M_{i2}\tilde{\delta}_2(t) = \mathcal{F}\left(\begin{bmatrix} 0 & U_{i2} \\ V_{i2} & 0 \end{bmatrix}, \tilde{\delta}_2(t) I_{r_{i2}}\right) \tag{A11}$$

2) An LFR of $M_i \delta_i(t)$ can be found as follows:

$$M_i \delta_i(t) = \mathcal{F}\left(\begin{bmatrix} 0 & [U_{i1} \quad 0] \\ \begin{bmatrix} 0 \\ V_{i2} \end{bmatrix} & \begin{bmatrix} 0 & V_{i1} U_{i2} \\ 0 & 0 \end{bmatrix} \end{bmatrix}, \mathrm{diag}\left(\tilde{\delta}_1(t) I_{r_{i1}}, \tilde{\delta}_2(t) I_{r_{i2}}\right)\right) \tag{A12}$$

For the sum of LFRs, it can be easily performed as

$$\mathcal{F}\left(\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \Delta_1\right) + \mathcal{F}\left(\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}, \Delta_2\right)$$

$$= \mathcal{F}\left(\begin{bmatrix} A_1 + A_2 & [B_1 \quad B_2] \\ \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} & \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \end{bmatrix}, \mathrm{diag}(\Delta_1, \Delta_2)\right) \tag{A13}$$

The procedure just outlined has many degrees of freedom, which can be exploited to optimize the computation burden of the synthesis procedure reducing the dimension of the matrix $\Delta(t)$.

Appendix B: HTF

Appendix B is dedicated to a concise presentation of the HTF. For additional details see Refs. 39 and 40.

Consider a continuous-time linear periodic system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad y(t) = C(t)x(t) + D(t)u(t) \tag{B1}$$

with $A(t+T) = A(t)$, $B(t+T) = B(t)$, $C(t+T) = C(t)$, and $D(t+T) = D(t)$. Each matrix can be expanded in a complex Fourier series

$$A(t) = \sum_{m \in Z} A_m e^{jm\Omega t} \tag{B2}$$

and similarly for $B(t)$, $C(t)$, and $D(t)$; the system can be analyzed in the frequency domain as follows.

Introduce the class of exponentially modulated periodic (EMP) signals.[39] The (complex) signal $u(t)$ is said to be EMP of period T and modulation s if

$$u(t) = \sum_{n \in Z} u_n e^{s_n t} \tag{B3}$$

with $t \geq 0$, $s_n = s + jn\Omega$, and $s \in C$.

The class of EMP signals is a generalization of the class of T-periodic signals: as a matter of fact, an EMP signal with $s = 0$ is just an ordinary time-periodic signal. In much the same way as a time-invariant system subject to a (complex) exponential input admits an exponential regime, a periodic system subject to an EMP input admits an EMP regime. In such a regime, all signals of interest can be expanded as EMP signals as follows:

$$x(t) = \sum_{n \in Z} x_n e^{s_n t} \tag{B4}$$

$$\dot{x}(t) = \sum_{n \in Z} s_n x_n e^{s_n t} \tag{B5}$$

$$y(t) = \sum_{n \in Z} y_n e^{s_n t} \tag{B6}$$

On the basis of expansion (B2) for $A(\cdot)$, and similar expansions for $B(\cdot)$, $C(\cdot)$, and $D(\cdot)$, the EMP regime response of the system can be expressed as an infinite-dimensional matrix equation of the following kind:

$$s\mathcal{X} = (\mathcal{A} - \mathcal{N})\mathcal{X} + \mathcal{B}\mathcal{U} \qquad \mathcal{Y} = \mathcal{C}\mathcal{X} + \mathcal{D}\mathcal{U} \tag{B7}$$

where $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ are doubly infinite vectors formed with the harmonics of x, u, and y, respectively, organized in the following fashion:

$$\mathcal{X}^T = \left[\ldots, x_{-2}^T, x_{-1}^T, x_0^T, x_1^T, x_2^T, \ldots\right] \tag{B8}$$

and similarly for $\mathcal{U}$and $\mathcal{Y}$.

$\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are doubly infinite Toeplitz matrices formed with the harmonics of $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, and $D(\cdot)$, respectively, as follows:

$$\mathcal{A} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & A_0 & A_{-1} & A_{-2} & A_{-3} & A_{-4} & \cdots \\ \cdots & A_1 & A_0 & A_{-1} & A_{-2} & A_{-3} & \cdots \\ \cdots & A_2 & A_1 & A_0 & A_{-1} & A_{-2} & \cdots \\ \cdots & A_3 & A_2 & A_1 & A_0 & A_{-1} & \cdots \\ \cdots & A_4 & A_3 & A_2 & A_1 & A_0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{B9}$$

and similarly for $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$. Matrix $\mathcal{N}$ is a block diagonal matrix

$$\mathcal{N} = \text{blkdiag}\{jn\Omega I\}, \qquad \forall n \in Z \tag{B10}$$

From Eq. (B7), one can define the HTF as the operator,

$$\hat{\mathcal{G}}(s) = \mathcal{C}[s\mathcal{I} - (\mathcal{A} - \mathcal{N})]^{-1}\mathcal{B} + \mathcal{D} \tag{B11}$$

Such an operator provides a most useful connection between the input harmonics and the output harmonics (organized in the infinite vectors $\mathcal{U}$and $\mathcal{Y}$, respectively).

In particular, if one takes $s = 0$, thus considering periodic regimes of the system, the appropriate input/output operator is

$$\hat{\mathcal{G}}(s) = \mathcal{C}[\mathcal{N} - \mathcal{A}]^{-1}\mathcal{B} + \mathcal{D} \tag{B12}$$

Such input/output representation of a periodic system is, however, impractical, given that it is infinite dimensional. From an engineering viewpoint, this model can be satisfactorily replaced by a finite-dimensional approximation obtained by truncation of the Fourier series of the system matrices, which in turn implies that matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{N}$ also are truncated and have finite dimensions.

Acknowledgments

This work was supported by Ministero dell Università e della Ricerca Scientifica e Tecnologica Project Identification and Control of Industrial Systems; CNR—Centro Studio Tecnologie dell'Informatica e dell'Automazione and Fondazione CARIPLO per la Ricerca Scientifica. Francesco A. Cuzzola has been partially supported by the Swiss National Science Foundation. The authors would like to thank Marco Lovera (Politecnico di Milano) and Roberto Celi (University of Maryland) for their helpful advice on helicopter control theory. The authors are also grateful to Carsten Scherer (Delft University) for the exchange of ideas on the subject of linear parameter varying control techniques and for his helpful suggestions.

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